Shear–Free Gravitational Waves in an Anisotropic Universe

P. A. Hogan[*] and E. M. O'Shea [†]

Mathematical Physics Department

National University of Ireland Dublin, Belfield, Dublin 4, Ireland

Abstract

We study gravitational waves propagating through an anisotropic Bianchi I dust–filled universe (containing the Einstein–de–Sitter universe as a special case). The waves are modeled as small perturbations of this background cosmological model and we choose a family of null hypersurfaces in this space–time to act as the histories of the wave–fronts of the radiation. We find that the perturbations we generate can describe pure gravitational radiation if and only if the null hypersurfaces are shear–free. We calculate the gauge–invariant small perturbations explicitly in this case. How these differ from the corresponding perturbations when the background space–time is isotropic is clearly exhibited.

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[*]E-mail : phogan@ollamh.ucd.ie

[†]E-mail : emer.oshea.2@student.ucd.ie

1 Introduction

In a recent paper [1] shear–free gravitational radiation analogous to the Bateman waves of electromagnetic theory was studied propagating through isotropic cosmological models. The radiation was represented as a perturbation of the cosmological model. It was analysed using the Ellis–Bruni [2] gauge–invariant and covariant perturbation formalism. Explicit solutions of the linear differential equations for the gauge–invariant small quantities (the Ellis–Bruni variables) were derived. These variables divide naturally into so–called vector quantities and tensor quantities. Examples of the vector quantities are the spatial gradients of the proper density of matter, of the isotropic pressure, of the expansion scalar of the matter world–lines, as well as the vorticity vector associated with the matter world–lines and the heat flow vector of the matter distribution. The tensor quantities include the shear of the matter world–lines and the anisotropic stress of the matter distribution. For perturbations describing gravitational waves the important variables needed are the tensor variables [1], [3]. If the cosmological model is isotropic then the shear of the matter world–lines and the anisotropic stress of the matter distribution are Ellis–Bruni variables. However if the cosmological model is *anisotropic* then the matter world–lines have shear and so the shear tensor is not (in its entirety at least) available to us as an Ellis–Bruni variable. This has important implications therefore for the study of gravitational wave propagation in an anisotropic universe. In this paper we show how to circumvent this difficulty. In addition we discover what influence the anisotropy in the universe has on the gravitational waves.

We are concerned here with deriving perturbations of an anisotropic cosmological model which can be interpreted as representing shear–free gravitational radiation. For definiteness we choose a Bianchi I dust–filled universe (containing the Einstein–de Sitter universe as a special case) and a simple family of null hypersurfaces in this space–time to act as the histories of the wave fronts of the radiation. To begin with these null hypersurfaces can have shear but we show that the perturbations we generate can only represent gravitational waves (with the gauge–invariant part of the perturbed Weyl tensor having the algebraic structure one associates with pure gravitational radiation) if the null hypersurfaces are shear–free. This involves a specialisation of the anisotropic universe. The perturbed quantities are explicitly calculated in this case and all are derived from a single complex–valued function whose dependence on one complex variable is analytic and arbitrary (a feature always associated with shear–free gravitational radiation) and whose dependence on two real variables is determined by a wave equation which can easily be integrated. If the background cosmological model is isotropic

(Einstein–de Sitter) then the wave equation is *second order* whereas if the background is anisotropic the wave equation is *first order*. This difference is directly attributable to the presence of shear in the matter world–lines in the anisotropic background.

Density perturbations of anisotropic Bianchi I universes have been thoroughly examined in the Ellis–Bruni formalism by Dunsby [4]. Our study of gravitational wave perturbations of a Bianchi I universe is complementary to this work and has in part been inspired by it.

The outline of the paper is as follows: The notation used throughout this paper and some important basic equations are given in Appendix A and referred to where appropriate. In Sec. II we describe the Bianchi I dust–filled universe which will play the role of a "background" space–time for our small perturbations. In Sec. III we identify the Ellis–Bruni gauge–invariant variables we shall use and give the equations satisfied by these variables. Then, in Sec. IV we require these Ellis–Bruni variables to have an arbitrary dependence on a function (in order to have the possibility of having gravitational waves with an arbitrary profile) and insert this dependence into the equations. The calculation of the perturbed quantities when the null hypersurfaces, described in the second paragraph of this introduction, are shear–free is also given in Sec. IV and the analogous calculations for the shearing hypersurfaces (also referred to above) are given in Appendix B. Finally, Sec. V is a discussion in which we summarise and comment on the results obtained in Sec. IV and in Appendix B.

2 An Anisotropic Space–Time

We are interested in constructing models of gravitational waves propagating through an anisotropic universe. For clarity we choose a specific Bianchi type I dust–filled universe which is the dust–filled counterpart of the Kasner universe (see, for example, [5]) . The space–time model has line–element

$$ds^2 = A^2(t)\,dx^2 + B^2(t)\,dy^2 + C^2(t)\,dz^2 - dt^2 \ , \tag{2.1}$$

with $A(t) = (t-k)^p(t+k)^{\frac{2}{3}-p}$, $B(t) = (t-k)^q(t+k)^{\frac{2}{3}-q}$, $C(t) = (t-k)^r(t+k)^{\frac{2}{3}-r}$, where p, q, r are constants satisfying $p+q+r = 1$ and $p^2+q^2+r^2 = 1$ and k is a constant (if $k = 0$ the line–element (2.1) becomes the Einstein–de–Sitter line–element).

The world–lines of the dust–particles are the time–like geodesic integral curves of the vector field $u^a\,\partial/\partial x^a = \partial/\partial t$ (thus $u^a = \delta^a_4$ since we shall label

the coordinates $x^1 = x,\ x^2 = y,\ x^3 = z$ and $x^4 = t$). The proper density is

$$\mu = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} + \frac{\dot{B}\dot{C}}{BC} = \frac{4}{3\,(t^2 - k^2)}\ , \tag{2.2}$$

where a dot indicates differentiation with respect to t. For future purposes it is useful to note that the field equations satisfied by the functions A, B, C given above are

$$\frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} = 0\,, \tag{2.3}$$

$$\frac{\dot{A}\dot{C}}{AC} + \frac{\ddot{A}}{A} + \frac{\ddot{C}}{C} = 0\,, \tag{2.4}$$

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} = 0. \tag{2.5}$$

The geodesic world–lines of the dust particles are twist–free (vorticity tensor $\omega_{ab} = 0$) and have *expansion scalar*

$$\theta = \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} = \frac{2t}{t^2 - k^2}\ , \tag{2.6}$$

and *shear tensor*

$$\sigma_{ab} = \sigma_{(1)} n_{(1)a} n_{(1)b} + \sigma_{(2)} n_{(2)a} n_{(2)b} + \sigma_{(3)} n_{(3)a} n_{(3)b}\ , \tag{2.7}$$

where $n_{(1)a} = A\,\delta^1_a\,,\ n_{(2)a} = B\,\delta^2_a\,,\ n_{(3)a} = C\,\delta^3_a$ and

$$\sigma_{(1)} = \frac{\dot{A}}{A} - \frac{1}{3}\,\theta\,, \qquad \sigma_{(2)} = \frac{\dot{B}}{B} - \frac{1}{3}\,\theta\,, \qquad \sigma_{(3)} = \frac{\dot{C}}{C} - \frac{1}{3}\,\theta\,, \tag{2.8}$$

with A, B, C given above and θ given by (2.6). The vorticity and shear tensors and the expansion scalar are defined in Appendix A.

We note that $\sigma_{ab} = \sigma_{ba}$ satisfies $\sigma^{ab}\,u_b = 0$ and, in addition, $\sigma^a{}_a = 0$ (this is equivalent to $\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} = 0$ which follows for example, from (2.6) and (2.8)). Following Ellis [6] it is convenient to define a *projected covariant derivative* operating on tensors which are orthogonal to u^a on all of their indices. In particular, if for example T_{abc} is such a tensor, then its projected covariant derivative, denoted by ${}^{(3)}\nabla_d\,T_{abc}$, is defined as

$${}^{(3)}\nabla_d\,T_{abc} = h^p_d\,h^q_a\,h^r_b\,h^s_c\,T_{qrs;p}\ , \tag{2.9}$$

where $h^a_b = \delta^a_b + u^a\,u_b$ is the projection tensor and the semi–colon indicates covariant differentiation. When this derivative operates on a scalar function

f it naturally specialises to the partial derivative projected orthogonal to u^a. Thus

$$ {}^{(3)}\nabla_a f = h_a^b \, f_{,b} \ , \tag{2.10} $$

with a comma denoting the partial derivative. We note that

$$ {}^{(3)}\nabla_a h_{bc} = 0 \, , \tag{2.11} $$

and thus the projected covariant derivative commutes with the operations of lowering and raising indices on tensors orthogonal to u^a using h_{ab} and h^{ab} respectively.

In the exact Bianchi type I universe the projected covariant derivative of any scalar function vanishes. Thus we have

$$ {}^{(3)}\nabla_a \theta = 0 \, , \qquad {}^{(3)}\nabla_a \mu = 0 \, . \tag{2.12} $$

Also

$$ {}^{(3)}\nabla_b \sigma_{(a)} = 0 \, , \tag{2.13} $$

for $a = 1, 2, 3$, and hence

$$ {}^{(3)}\nabla_b \sigma^2 = 0 \, , \tag{2.14} $$

where $\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab} = \sigma^2_{(1)} + \sigma^2_{(2)} + \sigma^2_{(3)}$. It follows from (2.7), (2.13) and the fact that ${}^{(3)}\nabla_c n^a_{(\alpha)} = 0$ with $\alpha = 1, 2, 3$ that

$$ {}^{(3)}\nabla_c \sigma_{ab} = 0. \tag{2.15} $$

With θ and σ_{ab} given by (2.6)and (2.7) respectively, it is easily shown using (2.3), (2.4), and (2.5) that

$$ \dot{\sigma}_{ab} + \theta \, \sigma_{ab} = 0. \tag{2.16} $$

Here and throughout a dot indicates covariant differentiation in the direction of u^a. Thus $\dot{\sigma}_{ab} = \sigma_{ab;c} u^a$.

With respect to the 4–velocity field defined after (2.1) the Weyl tensor with components C_{abcd}, is decomposed into an "electric part" and a "magnetic part" given by

$$ E_{ab} = C_{apbq} \, u^p \, u^q \, , \qquad H_{ab} = {}^*C_{apbq} \, u^p \, u^q \, . \tag{2.17} $$

Here ${}^*C_{apbq}$ is the dual of the Weyl tensor and it is defined by ${}^*C_{apbq} = \frac{1}{2} \eta_{ap}{}^{rs} \, C_{rsbq}$, where $\eta_{abcd} = \sqrt{-g} \, \epsilon_{abcd}$ with $g = \det(g_{ab})$ and ϵ_{abcd} is the Levi–Civita permutation symbol. The electric part of the Weyl tensor of the

anisotropic universe with line element (2.1) is non–zero and can be written in terms of the shear σ_{ab} of the matter world–lines as

$$E_{ab} = \frac{2}{3}\,\sigma^2\, h_{ab} - \sigma_{ac}\,\sigma^c{}_b + \frac{1}{3}\,\theta\,\sigma_{ab}\,. \tag{2.18}$$

This follows from the shear–propagation equation (see equation (3.5) below) in this space–time using the facts that the matter world–lines in this case are geodesic, twist–free with shear satisfying (2.16) and the dust matter distribution is stress–free. The magnetic part of the Weyl tensor vanishes in this universe and so we have

$$H_{ab} = 0. \tag{2.19}$$

In the rest of this paper we wish to construct perturbations of this anisotropic universe which we can interpret as gravitational waves propagating through the universe. The histories of the wave–fronts will be null hypersurfaces in this anisotropic (background) space–time. To be specific we shall choose the null hypersurfaces to have equations

$$\phi\,(x,t) \equiv\; x - T(t) =\; \text{constant}\,, \tag{2.20}$$

with $dT/dt = A^{-1}$. It is easy to see that these hypersurfaces are null with respect to the metric g_{ab} given via the line–element (2.1). Thus

$$g^{ab}\,\phi_{,a}\,\phi_{,b} = 0. \tag{2.21}$$

The integral curves of the vector field $g^{ab}\,\phi_{,b}$ are the null geodesic generators of the hypersurfaces (2.20). In terms of a convenient null tetrad in the anisotropic background given by the 1–forms,

$$\begin{aligned} k_a\,dx^a &= A\,dx - dt\;, \qquad l_a\,dx^a = -\frac{1}{2}(A\,dx + dt)\;, \\ m_a\,dx^a &= \frac{1}{\sqrt{2}}\,(B\,dy + i\,C\,dz)\;,\;\; \bar{m}_a\,dx^a = \frac{1}{\sqrt{2}}\,(Bdy - iCdz)\;, \end{aligned} \tag{2.22}$$

the complex shear of the null geodesics tangent to $\phi_{,a}$ is

$$\sigma\; = \phi_{,a;b}\,m^a\,m^b\; = \frac{1}{2A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right) = \frac{1}{2A}\,(\sigma_{(2)} - \sigma_{(3)})\;, \tag{2.23}$$

where we have used (2.8) in the last equality. We see that the null hypersurfaces are *shear–free* if and only if $B = C$. With the explicit expressions for A, B, C given following (2.1) we see that this corresponds to putting the constant $r = q$ and choosing q and p such that $p + 2\,q = 1$ and $p^2 + 2\,q^2 = 1$.

3 Equations for Perturbations

We wish to study gravitational wave propagation in the anisotropic universe described in Section II. We use the Ellis–Bruni approach [2] which involves working with small gauge–invariant and covariant quantities which vanish in the background rather than small perturbations of the background metric. The gauge–invariant variables used depend on the type of perturbations to be studied and also on the background space–time. For example, E_{ab} defined by (2.17) is an Ellis–Bruni variable for gravitational wave propagation in isotropic cosmologies [1], but in the present context its background value is non–zero (it is given by (2.18)) and hence it is not an Ellis–Bruni variable for our purposes. However, it is possible to extract a gauge–invariant part $\tilde{E}_{ab}$ of E_{ab} given by (see equations (3.13) and (3.14) below)

$$\tilde{E}_{ab} := \frac{1}{2}\,\pi_{ab} - (\dot{\sigma}_{ab} + \theta\,\sigma_{ab}). \tag{3.1}$$

Here π_{ab} is the small anisotropic stress in the perturbed matter distribution (see Appendix A). For our case the important Ellis–Bruni variables are $\tilde{E}_{ab}$, H_{ab}, π_{ab}, $\dot{\sigma}_{ab} + \theta\,\sigma_{ab}$ and ${}^{(3)}\nabla_c\,\sigma_{ab}$. We set all other Ellis–Bruni variables, including q_a, ω_{ab}, $h^b_c\,\theta_{,b}$, $h^b_c\,\mu_{,b}$, $h^b_c\,p_{,b}$ (p is the isotropic pressure) and $\dot{u}_a$, equal to zero since we found in [1] that it is tensor, not vector, quantities that describe gravitational wave perturbations (the vector quantities describe other types of perturbations such as inhomogeneities in the matter density). We now outline the equations satisfied by these variables.

When the Ricci identities,

$$u_{a;dc} - u_{a;cd} = R_{abcd}\,u^b\ , \tag{3.2}$$

where R_{abcd} is the Riemann curvature tensor, are projected along u^a and orthogonal to u^a (using the projection tensor) we obtain *Raychaudhuri's equation*,

$$\dot{\theta} + \frac{1}{3}\,\theta^2 - \dot{u}^a{}_{;a} + 2\,(\sigma^2 - \omega^2) + \frac{1}{2}(\mu + 3\,p) = 0\ , \tag{3.3}$$

(here $\omega^2 = \frac{1}{2}\,\omega^{ab}\,\omega_{ab}$) the *vorticity propagation equation*,

$$h^a_b\,\dot{\omega}^b + \frac{2}{3}\,\theta\,\omega^a = \sigma^a{}_b\,\omega^b + \frac{1}{2}\,\eta^{abcd}\,u_b\,\dot{u}_{c;d}\ , \tag{3.4}$$

(where $\omega^b := \frac{1}{2}\,\eta^{abcd}\,u_b\,\omega_{cd}$ is the vorticity vector) the *shear propagation equation*,

$$\begin{aligned} &h^f_a\,h^g_b\,(\dot{\sigma}_{fg} - \dot{u}_{(f;g)}) - \dot{u}_a\,\dot{u}_b + \omega_a\,\omega_b + \sigma_{af}\,\sigma^f{}_b + \frac{2}{3}\,\theta\,\sigma_{ab} \\ &+ h_{ab}\left(-\frac{1}{3}\,\omega^2 - \frac{2}{3}\,\sigma^2 + \frac{1}{3}\,\dot{u}^c{}_{;c}\right) - \frac{1}{2}\,\pi_{ab} + E_{ab} = 0\ , \end{aligned} \tag{3.5}$$

the *(0,ν)-field equation* (this terminology is explained in [6]),

$$\frac{2}{3}\,h^a_b\,\theta^{,b} - h^a_b\,\sigma^{bc}{}_{;d}\,h^d_c - \eta^{acdf}\,u_c\,(\omega_{d;f} + 2\,\omega_d\,\dot{u}_f) = q^a \ , \tag{3.6}$$

the *divergence of vorticity equation*,

$$\omega^a{}_{;b}\,h^b_a = \omega^a \dot{u}_a \ , \tag{3.7}$$

and the *magnetic part of the Weyl tensor*,

$$H_{ab} = 2\,\dot{u}_{(a}\,\omega_{b)} - h^t_a\,h^s_b\,(\omega_{(t}{}^{g;c} + \sigma_{(t}{}^{g;c})\,\eta_{s)fgc}\,u^f \ . \tag{3.8}$$

Next we consider the conservation equation,

$$T^{ab}{}_{;b} = 0 \ , \tag{3.9}$$

where T^{ab} is the energy-momentum-stress tensor (see Appendix A). This equation projected orthogonal to u^a and along u^a respectively gives the *equations of motion of matter*,

$$(\mu + p)\,\dot{u}^a + h^{ac}\,(p_{,c} + \pi^b{}_{c;b} + \dot{q}_c) + \left(\omega^{ab} + \sigma^{ab} + \frac{4}{3}\,\theta\,h^{ab}\right)\,q_b = 0 \ , \tag{3.10}$$

and the *energy conservation* equation,

$$\dot{\mu} + \theta\,(\mu + p) + \pi_{ab}\,\sigma^{ab} + q^a{}_{;a} + \dot{u}^a\,q_a = 0 \ . \tag{3.11}$$

Specialising equations (3.3)–(3.8) and (3.10)–(3.11) to the problem at hand, with perturbed quantities listed after (3.1) and q_a, ω_{ab}, $h^b_c\,\theta_{,b}$, $h^b_c\,\mu_{,b}$, $h^b_c\,p_{,b}$ and $\dot{u}_a$ *all put equal to zero*, we obtain the following equations: From *Raychaudhuri's equation*,

$$\dot{\theta} + \frac{1}{3}\,\theta^2 + 2\,\sigma^2 + \frac{1}{2}\,(\mu + 3\,p) = 0 \ . \tag{3.12}$$

From the *shear propagation equation*,

$$E_{ab} = \frac{1}{2}\,\pi_{ab} - \dot{\sigma}_{ab} - \sigma_{af}\,\sigma^f{}_b - \frac{2}{3}\,\theta\,\sigma_{ab} + \frac{2}{3}\,\sigma^2\,h_{ab} \ . \tag{3.13}$$

With $\tilde{E}_{ab}$ given by (3.1) we can write this equation as

$$E_{ab} = \tilde{E}_{ab} - \sigma_{af}\,\sigma^f{}_b + \frac{1}{3}\,\theta\,\sigma_{ab} + \frac{2}{3}\,\sigma^2\,h_{ab} \ . \tag{3.14}$$

We now see explicitly from equations (2.19) and (3.14) that $\tilde{E}_{ab} = 0$ in the anisotropic background. This justifies the definition of $\tilde{E}_{ab}$ in equation (3.1).

From the *(0,ν)-field equation*

$$ {}^{(3)}\nabla_a \, \sigma^{ab} = 0. \tag{3.15} $$

From the *magnetic part of the Weyl tensor*

$$ H_{ab} = -\, h^s_{(a} \, {}^{(3)}\nabla^c \, \sigma^p{}_{b)} \, \eta_{sfpc} \, u^f \ . \tag{3.16} $$

From the *equations of motion of matter*

$$ h^{ac} \, \pi^b{}_{c;b} = 0 \ . \tag{3.17} $$

Finally the energy conservation equation gives

$$ \dot{\mu} + \theta \, (\mu + p) + \pi_{ab} \, \sigma^{ab} = 0 \ . \tag{3.18} $$

The remaining two equations derived from the Ricci identities (the vorticity propagation equation and the divergence of vorticity equation) are identically satisfied. We note that equations (3.12) and (3.18) are not expressed in terms of the Ellis–Bruni variables we wish to work with and so are not immediately useful. However, taking the projected covariant derivative of these equations and retaining only linear terms yields: *the projected covariant derivative of Raychaudhuri's equation*,

$$ \sigma_{ab} \, {}^{(3)}\nabla_c \, \sigma^{ab} = 0 \ , \tag{3.19} $$

and *the projected covariant derivative of the energy conservation equation*,

$$ \sigma^{ab} \, {}^{(3)}\nabla_c \, \pi_{ab} = 0 \ . \tag{3.20} $$

When the Bianchi identities, written conveniently in the form

$$ C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6} \, g^{c[a} \, R^{;b]} \ , \tag{3.21} $$

where $R^{ca} := R^{cba}{}_b$ are the components of the Ricci tensor and $R := R^c{}_c$ is the Ricci scalar, are projected along u^a and orthogonal to u^a we obtain equations for E_{ab} and H_{ab} which are analogous to Maxwell's equations. The general forms of these equations are lengthy and to keep this section to a reasonable length we give them in Appendix A. In terms of the perturbed quantities given following (3.1) and with q_a, ω_{ab}, $h^b_c \, \theta_{,b}$, $h^b_c \, \mu_{,b}$, $h^b_c \, p_{,b}$ and $\dot{u}_a$ all vanishing these equations are: the *div–E equation* (using (3.17)),

$$ {}^{(3)}\nabla_a \, \tilde{E}^{ab} = \sigma^{af} \, {}^{(3)}\nabla_a \, \sigma^b{}_f + \eta^{bapq} \, u_a \, \sigma^d{}_p \, H_{qd} \ , \tag{3.22} $$

the *div–H equation*,

$$ {}^{(3)}\nabla_a\, H^{ab} = -\,\eta^{bapq}\, u_a\, \sigma^d{}_p\, \left(\tilde{E}_{qd} + \frac{1}{2}\,\pi_{qd}\right)\ , \tag{3.23} $$

the $\dot{H}$–*equation*,

$$ \begin{aligned} &\dot{H}^{bt} - h^{(b}_a\, \eta^{t)rsd}\, u_r\, \tilde{E}^a{}_{s;d} - \eta^{rsd(t}\, u_r\, {}^{(3)}\nabla_d\, \sigma^{b)f}\, \sigma_{fs} - \eta^{rsd(t}\, u_r\, \sigma^{b)f}\, {}^{(3)}\nabla_d\, \sigma_{fs} \\ &-\frac{1}{3}\,\theta\, \left({}^{(3)}\nabla_d\, \sigma^{(b}{}_s\right)\, \eta^{t)rsd}\, u_r - 3\, H^{(t}{}_s\, \sigma^{b)s} + h^{bt}\, H^{dp}\, \sigma_{dp} + \theta\, H^{bt} \\ &= -\,\frac{1}{2}\,\eta^{(b}{}_{rad}\, \pi^{t)a;d}\, u^r + \frac{1}{2}\,\eta^{(b}{}_{rad}\, u^{t)}\, u^r \sigma^{cd}\, \pi^a{}_c\ , \end{aligned} \tag{3.24} $$

and the $\dot{E}$–*equation*,

$$ \begin{aligned} &\left(\frac{1}{3}\,\dot{\theta} - 2\,\sigma^2 + \frac{1}{2}\,\mu\right)\, \sigma^{bt} + 3\,\sigma^b{}_f\, \sigma^f{}_s\, \sigma^{st} - h^{bt}\, \sigma^a{}_f\, \sigma^{fd}\, \sigma_{da} \\ &= -\dot{\tilde{E}}^{bt} - \frac{2}{3}\,\sigma_{fg}\, (\dot{\sigma}^{fg} + \theta\, \sigma^{fg})\, h^{bt} - \theta\, \tilde{E}^{bt} + (\dot{\sigma}^b{}_f + \theta\, \sigma^b{}_f)\, \sigma^{ft} \\ &+(\dot{\sigma}^{ft} + \theta\, \sigma^{ft})\, \sigma^b{}_f - \frac{1}{3}\,\theta\, (\dot{\sigma}^{bt} + \theta\, \sigma^{bt}) - h^{(b}_a\, \eta^{t)rsd}\, u_r\, H^a{}_{s;d} \\ &+3\, \tilde{E}^{(t}{}_c\, \sigma^{b)c} - h^{bt}\, \tilde{E}^{dp}\, \sigma_{dp} + \frac{1}{6}\, h^{bt}\, \pi^{dp}\, \sigma_{dp} - \frac{1}{2}\,\dot{\pi}^{bt} - \frac{1}{2}\sigma^{c(b}\, \pi^{t)}{}_c \\ &-\frac{1}{6}\,\theta\, \pi^{bt}\ . \end{aligned} \tag{3.25} $$

We note that the right–hand side of equation (3.25) is expressed in terms of Ellis–Bruni variables and thus the left hand side must be an Ellis–Bruni variable which we denote by

$$ W^{bt} := \left(\frac{1}{3}\,\dot{\theta} - 2\,\sigma^2 + \frac{1}{2}\,\mu\right)\, \sigma^{bt} + 3\,\sigma^b{}_f\, \sigma^f{}_s\, \sigma^{st} - h^{bt}\, \sigma^a{}_f\, \sigma^{fd}\, \sigma_{ad}\ . \tag{3.26} $$

Calculating the projected covariant derivative of W^{bt} gives

$$ \begin{aligned} {}^{(3)}\nabla_c\, W^{bt} &= -3\,\sigma^2\, {}^{(3)}\nabla_c\, \sigma^{bt} + 3\, {}^{(3)}\nabla_c\, (\sigma^b{}_f\, \sigma^f{}_s\, \sigma^{st}) \\ &\quad -h^{bt}\, {}^{(3)}\nabla_c(\sigma^a{}_f\, \sigma^{fd}\, \sigma_{da})\ , \end{aligned} \tag{3.27} $$

where we have used the background values of θ, μ and σ^{ab} given in the previous section to simplify the coefficient of ${}^{(3)}\nabla_c\, \sigma^{bt}$. Using (3.25) and (3.1) we can express W^{tb} in terms of $\tilde{E}^{bt}$, H^{bt} and π^{bt} as

$$ \begin{aligned} W^{bt} &= -\dot{\tilde{E}}^{bt} - \frac{2}{3}\,\theta\, \tilde{E}^{bt} - \frac{1}{2}\,\dot{\pi}^{bt} - \frac{1}{3}\,\theta\, \pi^{bt} - \frac{1}{6}\, h^{bt}\, \pi^{dp}\, \sigma_{dp} \\ &-\frac{1}{3}\, h^{bt}\, \tilde{E}^{dp}\, \sigma_{dp} + \frac{1}{2}\, \sigma^{(b}{}_f\, \pi^{t)f} + \tilde{E}^{(t}{}_f\, \sigma^{b)f} - h^{(b}_a\, \eta^{t)rsd}\, u_r\, H^a{}_{s;d}\ . \end{aligned} \tag{3.28} $$

We shall use this expression later in the left–hand side of (3.27).

Next we examine Eq. (3.23). Substituting for H^{ab} from (3.16) gives

$$\eta^{frsd}\,u_r\left\{-\frac{1}{2}\left({}^{(3)}\nabla_d\,\sigma^b{}_s\right)_{;\,f}+u^b\,\sigma_{fa}\,{}^{(3)}\nabla_d\,\sigma^a{}_s\right\}-\frac{1}{2}\,\eta^{brsd}\,\sigma_{ra}\,{}^{(3)}\nabla_d\,\sigma^a{}_s$$
$$-\eta^{brsd}\,u_r\left\{\frac{1}{2}\left({}^{(3)}\nabla_d\,\sigma^a{}_s\right)_{;\,a}-\sigma^a{}_s\left(\tilde{E}_{da}+\frac{1}{2}\,\pi_{ad}\right)\right\}=0\ . \tag{3.29}$$

Multiplying across by η_{blmq} and using

$$u_b\left({}^{(3)}\nabla_d\,\sigma^b{}_s\right)_{;\,l}=-\frac{1}{3}\,\theta\,{}^{(3)}\nabla_d\,\sigma_{ls}\,-\,\sigma_{bl}\,{}^{(3)}\nabla_d\,\sigma^b{}_s\ , \tag{3.30}$$

(since ${}^{(3)}\nabla_d\,\sigma^b{}_s$ is orthogonal to u_b) and

$$\left({}^{(3)}\nabla_m\,\sigma^b{}_q\right)_{;\,b}-u_m\,\sigma^{ab}\,{}^{(3)}\nabla_a\,\sigma_{bq}={}^{(3)}\nabla_b\left({}^{(3)}\nabla_m\,\sigma^b{}_q\right)\ , \tag{3.31}$$

we find that the $div-H$ equation implies that

$${}^{(3)}\nabla_b\left({}^{(3)}\nabla_{[m}\,\sigma^b{}_{q]}\right)+\sigma^a{}_{[m}\left(\tilde{E}_{q]a}+\frac{1}{2}\,\pi_{q]a}\right)=0\ , \tag{3.32}$$

with the square brackets as always denoting skew–symmetrisation. The converse is also true. This is shown by multiplying (3.32) by η^{rsmq} and using

$${}^{(3)}\nabla_m\,\sigma^b{}_q-{}^{(3)}\nabla_q\,\sigma^b{}_m=\eta_{qpfm}\,u^p\,H^{bf}\ , \tag{3.33}$$

which is easily derived from (3.16). Thus we can replace (3.23) by (3.32) without any loss of information.

The projected covariant derivative and the covariant derivative in the direction of u^a do not commute. The commutation relation satisfied by these derivatives when acting on σ_{ab} follows from the Ricci identities and reads:

$${}^{(3)}\nabla_c\,\dot{\sigma}_{ab}-\left({}^{(3)}\nabla_c\,\sigma_{ab}\right)^{\cdot}=\sigma^s{}_a\,{}^{(3)}\nabla_b\,\sigma_{cs}-\sigma^s{}_a\,{}^{(3)}\nabla_s\,\sigma_{cb}+\sigma^s{}_b\,{}^{(3)}\nabla_a\,\sigma_{cs}$$
$$-\sigma^s{}_b\,{}^{(3)}\nabla_s\,\sigma_{ca}+\sigma^s{}_c\,{}^{(3)}\nabla_s\,\sigma_{ab}+\frac{1}{3}\,\theta\,{}^{(3)}\nabla_c\,\sigma_{ab}\ . \tag{3.34}$$

In the present context this important equation can be written solely in terms of gauge–invariant variables. Thus we shall use it in the form,

$${}^{(3)}\nabla_c\,(\,\dot{\sigma}_{ab}+\theta\,\sigma_{ab}\,)=\left({}^{(3)}\nabla_c\,\sigma_{ab}\right)^{\cdot}-\sigma^f{}_b\,{}^{(3)}\nabla_f\,\sigma_{ac}-\sigma^f{}_a\,{}^{(3)}\nabla_f\,\sigma_{bc}$$
$$+\sigma^f{}_b\,{}^{(3)}\nabla_a\,\sigma_{fc}+\sigma^f{}_a\,{}^{(3)}\nabla_b\,\sigma_{fc}+\sigma^f{}_c\,{}^{(3)}\nabla_f\,\sigma_{ab}+\frac{4}{3}\,\theta\,{}^{(3)}\nabla_c\,\sigma_{ab}\ . \tag{3.35}$$

Using this equation and (3.1) we obtain the following equation for the projected covariant derivative of $\tilde{E}_{ab}$,

$$ {}^{(3)}\nabla_c\,\tilde{E}_{ab} = \frac{1}{2}\,{}^{(3)}\nabla_c\,\pi_{ab} - \left({}^{(3)}\nabla_c\,\sigma_{ab}\right)^{\cdot} + \sigma^f{}_b\,{}^{(3)}\nabla_f\,\sigma_{ac} + \sigma^f{}_a\,{}^{(3)}\nabla_f\,\sigma_{bc} $$
$$ -\sigma^f{}_b\,{}^{(3)}\nabla_a\,\sigma_{fc} - \sigma^f{}_a\,{}^{(3)}\nabla_b\,\sigma_{fc} - \sigma^f{}_c\,{}^{(3)}\nabla_f\,\sigma_{ab} - \frac{4}{3}\,\theta\,{}^{(3)}\nabla_c\,\sigma_{ab}\ . \quad (3.36) $$

Contracting this equation over a and c and making use of (3.15) and (3.17) gives

$$ {}^{(3)}\nabla_a\,\tilde{E}^{ab} = 0\ . \quad (3.37) $$

With H_{ab} given by (3.16) it is easily shown that (3.22) is consistent with this equation. It also follows from (3.36), (3.19) and (3.20) that

$$ \sigma^{ab}\,{}^{(3)}\nabla_c\,\tilde{E}_{ab} = 0\ . \quad (3.38) $$

The equations we will use are (3.1), (3.14), (3.15), (3.16), (3.17), (3.19), (3.20), (3.24), (3.27), (3.28), (3.32), (3.35), (3.37) and (3.38). These equations are not independent of each other. For example (3.24) is automatically satisfied and this can be seen as follows: First using (3.13) and (3.35) we show that

$$ h^b_a\,\eta^{trsd}\,u_r\,\left(E^a{}_s - \frac{1}{2}\,\pi^a{}_s\right)_{;\,d} = -\,\eta^{trsd}\,u_r\,\left({}^{(3)}\nabla_d\,\sigma^b{}_s\right)^{\cdot} - \theta\,\eta^{trsd}\,u_r\,{}^{(3)}\nabla_d\,\sigma^b{}_s $$
$$ -\eta^{trsd}\,u_r\,\sigma^f{}_s\,(\,{}^{(3)}\nabla^b\,\sigma_{fd} - {}^{(3)}\nabla_f\,\sigma^b{}_d + {}^{(3)}\nabla_d\,\sigma^b{}_f - {}^{(3)}\nabla_f\,\sigma^b{}_d). \quad (3.39) $$

Then substituting for ${}^{(3)}\nabla_d\,\sigma^b{}_f - {}^{(3)}\nabla_f\,\sigma^b{}_d$ from (3.33) in this equation we find that

$$ h^{(b}_a\,\eta^{t)rsd}\,u_r\,\left(E^a{}_s - \frac{1}{2}\,\pi^a{}_s\right)_{;\,d} = \dot{H}^{bt} - 3\,\sigma^{(t}{}_d\,H^{b)d} + h^{bt}\,\sigma^{dp}\,H_{dp} - \theta\,H^{bt}. \quad (3.40) $$

If we replace $E^a{}_s$ in this equation with the right-hand side of (3.14) the result is Eq. (3.24) and thus the $\dot{H} - \textit{equation}$ is automatically satisfied. We shall demonstrate the internal consistencies of the remaining equations by finding a solution which satisfies all the equations.

4 Shear–Free Gravitational Waves

We shall now look for solutions ${}^{(3)}\nabla_c\,\sigma_{ab}$, $\dot{\sigma}_{ab} + \theta\,\sigma_{ab}$ and π_{ab} of the equations given in the previous section for which these variables depend upon

an arbitrary function. We expect that this dependence of perturbations will describe gravitational waves carrying arbitrary information. Specifically we assume that

$$\begin{aligned} {}^{(3)}\nabla_c\,\sigma_{ab} &= A_{cab}\,F(\phi) + G_{cab}\,F'(\phi)\ , \\ \dot{\sigma}_{ab} + \theta\,\sigma_{ab} &= B_{ab}\,F(\phi) + C_{ab}\,F'(\phi)\ , \\ \pi_{ab} &= \Pi_{ab}\,F(\phi)\ , \end{aligned} \tag{4.1}$$

where F is an arbitrary real–valued function of its argument $\phi(x^a)$ and F' denotes the derivative of F with respect to its argument. This idea of introducing arbitrary functions into solutions of Einsteins equations describing gravitational waves goes back to work by Trautman [7]. Its use in the context of gauge–invariant perturbations of cosmological models was initiated by Hogan and Ellis [3] and further developed by Hogan and O'Shea [1]. We note that all of the quantities in (4.1) are orthogonal to u^a and B_{ab}, C_{ab} and Π_{ab} are trace–free with respect to the background metric g_{ab} (i.e. $B^a{}_a = 0$, $C^a{}_a = 0$ and $\Pi^a{}_a = 0$).

When we substitute the first two equations in Eq. (4.1) into the commutation relation (3.35) we, in effect, obtain integrability conditions to be satisfied by the right hand sides of these two equations. Remembering that F is an *arbitrary* function this substitution results in the following equations:

$$C_{ab}\,\lambda_c = G_{cab}\,\dot{\phi}\ , \tag{4.2}$$

where $\lambda_c = h^b_c\,\phi_{,b}$ and $\dot{\phi} = \phi_{,a}u^a$,

$$\begin{aligned} B_{ab}\,\lambda_c + {}^{(3)}\nabla_c\,C_{ab} &= A_{cab}\,\dot{\phi} + \dot{G}_{cab} - \sigma^f{}_b\,G_{fac} - \sigma^f{}_a\,G_{fbc} \\ &\quad + \sigma^f{}_b\,G_{afc} + \sigma^f{}_a\,G_{bfc} + \sigma^f{}_c\,G_{fab} + \frac{4}{3}\,\theta\,G_{cab}\ , \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} {}^{(3)}\nabla_c\,B_{ab} &= \dot{A}_{cab} - \sigma^f{}_b\,A_{fac} - \sigma^f{}_a\,A_{fbc} + \sigma^f{}_b\,A_{afc} \\ &\quad + \sigma^f{}_a\,A_{bfc} + \sigma^f{}_c\,A_{fab} + \frac{4}{3}\,\theta\,A_{cab}\ . \end{aligned} \tag{4.4}$$

If we now let

$$C_{ab} = \dot{\phi}\,s_{ab}\ , \tag{4.5}$$

where $s_{ab}\,u^a = 0$ and $s^a{}_a = 0$ then it immediately follows from (4.2) that

$$G_{cab} = s_{ab}\,\lambda_c\ . \tag{4.6}$$

As a result of (4.5) and (4.6) and also since

$$ {}^{(3)}\nabla_c \dot{\phi} = (\phi_{,c})^{\cdot} + \ddot{\phi}\, u_c + \frac{1}{3}\,\theta\,\lambda_c + \sigma^p{}_c\,\lambda_p \ , \tag{4.7} $$

and

$$ (\lambda_c)^{\cdot} = (\phi_{,c})^{\cdot} + \ddot{\phi}\, u_c \ , \tag{4.8} $$

(4.3) now reads

$$ \begin{aligned} B_{ab}\,\lambda_c - A_{cab}\,\dot{\phi} \qquad &= -\dot{\phi}\,{}^{(3)}\nabla_c\, s_{ab} - \frac{1}{3}\,\theta\,\lambda_c\, s_{ab} + \dot{s}_{ab}\,\lambda_c - \sigma^f{}_b\,\lambda_f\, s_{ac} \\ &\quad -\sigma^f{}_a\,\lambda_f\, s_{bc} + \sigma^f{}_b\,\lambda_a\, s_{fc} + \sigma^f{}_a\,\lambda_b\, s_{fc} + \frac{4}{3}\,\theta\,\lambda_c\, s_{ab} \ . \end{aligned} \tag{4.9} $$

Substituting Eqs. (4.1) (with C_{ab} and G_{cab} replaced by (4.5) and (4.6) respectively) into Eqs. (3.1), (3.15), (3.16) (3.17), (3.19), (3.20), (3.27), (3.28), (3.32), (3.37) and (3.38) results in the following list of equations: From the *(0, ν) field equation*,

$$ s^{ab}\,\lambda_b = 0 \ , \tag{4.10} $$

$$ A_b{}^{ab} = 0 \ . \tag{4.11} $$

From the *magnetic part of the Weyl tensor*,

$$ H_{ab} = q_{ab}\, F + l_{ab}\, F' \ , \tag{4.12} $$

where as always $F' = dF/d\phi$ and

$$ q_{ab} = -\, A^{cp}{}_{(\,b}\,\eta_{a)fpc}\, u^f \ , \tag{4.13} $$

$$ l_{ab} = -\, \lambda^c\, s^p{}_{(\,b}\,\eta_{a)fpc}\, u^f \ . \tag{4.14} $$

From the *equations of motion of matter*,

$$ \Pi^{ab}{}_{;b} = u^a\,\pi^{bc}\,\sigma_{bc} \ , \tag{4.15} $$

$$ \Pi^{ab}\,\phi_{,b} = 0 \ . \tag{4.16} $$

From the *projected covariant derivative of Raychaudhuri's equation*,

$$ \sigma^{ab}\, A_{cab} = 0 \ , \tag{4.17} $$

$$ \sigma^{ab}\, s_{ab} = 0 \ . \tag{4.18} $$

From the *projected covariant derivative of the energy conservation equation*,

$$ \sigma_{ab}\,{}^{(3)}\nabla_c\,\Pi^{ab} = 0 \ , \tag{4.19} $$

$$\sigma_{ab}\,\Pi^{ab} = 0\ . \tag{4.20}$$

From (3.1)

$$\tilde{E}_{ab} = (\frac{1}{2}\,\Pi_{ab} - B_{ab})\,F - \dot{\phi}\,s_{ab}\,F'\ . \tag{4.21}$$

From (3.32)

$$\begin{aligned} &{}^{(3)}\nabla_b\,A_m{}^b{}_q - {}^{(3)}\nabla_b\,A_q{}^b{}_m + \sigma^a{}_m\,(\Pi_{qa} - B_{qa}) \\ &-\sigma^a{}_q(\Pi_{ma} - B_{ma}) = 0\ , \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} &A_m{}^b{}_q\,\lambda_b + \lambda_m\,{}^{(3)}\nabla_b\,s^b{}_q + s^b{}_q\,{}^{(3)}\nabla_b\,\lambda_m - A_q{}^b{}_m\,\lambda_b - \lambda_q{}^{(3)}\nabla_b\,s^b{}_m \\ &-s^b{}_m\,{}^{(3)}\nabla_b\,\lambda_q - \dot{\phi}\,\sigma^a{}_m\,s_{qa} + \dot{\phi}\,\sigma^a{}_q\,s_{ma} = 0\ . \end{aligned} \tag{4.23}$$

From (3.37)

$${}^{(3)}\nabla_a\,B_{ab} = 0\ , \tag{4.24}$$

$$B^{ab}\,\lambda_a + \dot{\phi}\,{}^{(3)}\nabla_a\,s^{ab} + s^{ab}\,{}^{(3)}\nabla_a\,\dot{\phi} = 0\ , \tag{4.25}$$

$$s^{ab}\,\lambda_b = 0\ . \tag{4.26}$$

From (3.38)

$$\sigma_{ab}\,{}^{(3)}\nabla_c\,B^{ab} = 0\ , \tag{4.27}$$

$$\sigma_{ab}\,B^{ab} = 0\ . \tag{4.28}$$

From (3.27)

$$\begin{aligned} {}^{(3)}\nabla_c\,W^{bt} \;=\;& 3\,(-\sigma^2\,A_c{}^{bt} + \sigma^b{}_f\,\sigma^f{}_s\,A_c{}^{st} + \sigma^b{}_f\,\sigma^{st}\,A_c{}^f{}_s + \sigma^f{}_s\sigma^{st}\,A_c{}^b{}_f \\ &-h^{bt}\,\sigma^d{}_a\,\sigma^{af}\,A_{cfd})\,F + 3(-\sigma^2\,s^{bt} + \sigma^b{}_f\,\sigma^f{}_s\,s^{st} + \sigma^b{}_f\,\sigma^{st}\,s^f{}_s \\ &+\sigma^f{}_s\,\sigma^{st}\,s^b{}_f - h^{bt}\,\sigma^d{}_a\,\sigma^{af}\,s_{fd})\,\lambda_c\,F'\ . \end{aligned} \tag{4.29}$$

Finally, from Eq. (3.28) using (4.12)–(4.14), (4.18), (4.20), (4.21) and (4.28) we find that

$$\begin{aligned} W^{bt} \;=\;& \{\dot{\Pi}^{bt} - \frac{2}{3}\,\theta\,\Pi^{bt} + \dot{B}^{bt} + \frac{2}{3}\,\theta\,B^{bt} + \sigma^{(b}{}_f\,\Pi^{t)f} - B^{(t}{}_f\,\sigma^{b)f} \\ &{}^{(3)}\nabla_d\,(A^{dbt} - \frac{1}{2}\,A^{tdb} - \frac{1}{2}\,A^{bdt})\}F + \{(B^{bt} - \Pi^{bt} + \dot{s}^{bt} \\ &+\frac{2}{3}\,\theta\,s^{bt} - s^{(t}{}_f\,\sigma^{b)f})\,\dot{\phi} + \ddot{\phi}\,s^{bt} - A^{dbt}\,\lambda_d - \lambda^d\,{}^{(3)}\nabla_d\,s^{bt} \\ &-s^{bt}\,{}^{(3)}\nabla_d\,\lambda^d + \frac{1}{2}\,A^{tdb}\,\lambda_d + \frac{1}{2}\,\lambda^t\,{}^{(3)}\nabla_d\,s^{db} + \frac{1}{2}\,s^{db}\,{}^{(3)}\nabla_d\,\lambda^t \\ &+\frac{1}{2}\,A^{bdt}\,\lambda_d + \frac{1}{2}\,s^{dt}\,{}^{(3)}\nabla_d\,\lambda^b + \frac{1}{2}\,\lambda^b\,{}^{(3)}\nabla_d\,s^{dt}\}F' + \\ &\{\dot{\phi}^2\,s^{bt} - \lambda^d\,\lambda_d\,s^{bt}\}F''\ . \end{aligned} \tag{4.30}$$

Replacing W^{bt} in (4.29) by the right–hand side of (4.30) and equating the coefficients of F''' yields:

$$(\dot{\phi}^2\, s^{bt} - \lambda^d\, \lambda_d\, s^{bt})\, \lambda_c = 0\ . \tag{4.31}$$

Assuming $\lambda_c \neq 0$ this implies that

$$\phi_{,a}\, \phi^{,a} = 0\ . \tag{4.32}$$

Thus the hypersurfaces $\phi\,(x^a)$ = constant in the background anisotropic cosmological model are null. The equations found here by equating the coefficients of F, F' and F'' are extremely complicated and we will work with them in a completely different way below.

For the remainder of this section we seek to construct perturbations of the anisotropic background cosmological model which describe *shear–free* gravitational waves having the null hypersurfaces $\phi(x^a)$ = constant as the histories of their wave fronts in the background space–time. For the function $\phi(x^a)$ in (4.32) we shall take $\phi(x,t)$ given by (2.20) and in order to have these null hypersurfaces shear–free we see from (2.23) that we must have $\sigma_{(2)} = \sigma_{(3)}$. This is a relationship between the (non–vanishing) principal shears of the background matter distribution. One immediate consequence of this is that $\tilde{E}_{ab}$ is no longer the *full* gauge invariant part of E_{ab}. To see this we note that in a general space–time with metric g_{ab} and a preferred congruence of world–lines tangent to u^a we can write the shear tensor σ_{ab} in terms of the principal shears $\sigma_{(1)}$, $\sigma_{(2)}$, $\sigma_{(3)}$ as

$$\sigma_{ab} = \sigma_{(1)}\, n_{(1)\,a}\, n_{(1)\,b} + \sigma_{(2)}\, n_{(2)\,a}\, n_{(2)\,b} + \sigma_{(3)}\, n_{(3)\,a}\, n_{(3)\,b}\ , \tag{4.33}$$

where $\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} = 0$ and $n_{(1)\,a}$, $n_{(2)\,a}$ and $n_{(3)\,a}$ are the unit orthogonal (space–like) eigenvectors of σ_{ab}. Then letting $m^a = (n^a_{(2)} + i\, n^a_{(3)})/\sqrt{2}$ and using (4.33) we can write

$$\left(\frac{1}{3}\,\theta\,\sigma_{ab} - \sigma_{af}\,\sigma^f{}_b - \frac{2}{3}\,\sigma^2\,h_{ab}\right)\, m^a\, m^b = \left(\sigma_{(1)} + \frac{1}{3}\,\theta\right)\,\sigma_{ab}\, m^a\, m^b\ , \tag{4.34}$$

where

$$\sigma_{ab}\, m^a\, m^b = \frac{1}{2}\,(\sigma_{(2)} - \sigma_{(3)})\ , \tag{4.35}$$

is now a gauge–invariant variable which vanishes in the background space–time in which $\sigma_{(2)} = \sigma_{(3)}$. When σ_{ab} is the perturbed shear m^a is given its background value which, for our case, is given implicitly in (2.22). Thus we can give $\sigma_{(1)} + \frac{1}{3}\,\theta$ its background value when (4.34) is calculated in first approximation. It follows from this and (3.14) that now in first approximation

the full gauge invariant part of E_{ab} is

$$E'_{ab} := \tilde{E}_{ab} + \frac{\dot{A}}{A}\,\sigma_{rs}\,m^r\,m^s\,\bar{m}_a\,\bar{m}_b + \frac{\dot{A}}{A}\,\sigma_{rs}\,\bar{m}^r\,\bar{m}^s\,m_a\,m_b\ . \tag{4.36}$$

We shall consider pure pure gravity wave perturbations of the anisotropic background. We do this by requiring that the *gauge–invariant* perturbed "magnetic" and "electric" parts (H_{ab} and E'_{ab} respectively) of the Weyl tensor be type N in the Petrov classification with $\phi^{,a}$ as degenerate principal null direction. We achieve this provided $E'_{ab}\,\phi^{,b} = 0 = H_{ab}\,\phi^{,b}$. This means that, in light of (4.10) and (4.16), we should require

$$B^{ab}\,\phi_{,b} = 0\ , \qquad l^{ab}\,\phi_{,b} = 0\ , \qquad q^{ab}\,\phi_{,b} = 0\ , \tag{4.37}$$

with l^{ab} , q^{ab} given above in (4.13) and (4.14). The first of these allows us to write Eq. (4.25) as

$$^{(3)}\nabla_a\,(\dot{\phi}\,s^{ab}) = 0\ , \tag{4.38}$$

which we can simplify to

$$(\dot{\phi}\,s^{ab})_{;b} = 0\ . \tag{4.39}$$

We note that it follows from the definition of l_{ab} given in (4.14) that the second of (4.37) is identically satisfied.

Since s^{ab}, Π^{ab} and B^{ab} are orthogonal to u^a and $\phi^{,a}$ and trace–free with respect to the metric tensor given via the line–element (2.1) each have only two independent components, $-s^{22} = s^{33} = \alpha\,(x,\,y,\,z,\,t\,)$, $s^{23} = s^{32} = \beta\,(x,\,y,\,z,\,t\,)$, $B^{23} = B^{32}$, $-B^{22} = B^{33}$, $\Pi^{23} = \Pi^{32}$ and $-\Pi^{22} = \Pi^{33}$. It follows from this and the fact that $\sigma_{(2)} = \sigma_{(3)}$ that (4.18), (4.20) and (4.28) (and hence (4.19) and (4.27)) are identically satisfied.

Calculation of (4.39) shows that α, β must satisfy the Cauchy–Riemann equations,

$$\frac{\partial\,\alpha}{\partial\,y} - \frac{\partial\,\beta}{\partial\,z} = 0\ , \tag{4.40}$$

$$\frac{\partial\,\beta}{\partial\,y} + \frac{\partial\,\alpha}{\partial\,z} = 0\ . \tag{4.41}$$

We can write these equations economically as

$$\frac{\partial}{\partial\,\bar{\zeta}}(\alpha + i\,\beta) = 0\ , \tag{4.42}$$

with $\zeta = y + i\,z$. In addition, combining (4.15), (4.20), (4.24) and (4.27) we find that Π^{ab} and B^{ab} must also satisfy the Cauchy–Riemann equations. Thus we have,

$$\frac{\partial\,\Pi^{33}}{\partial\,y} - \frac{\partial\,\Pi^{23}}{\partial\,z} = 0\ , \tag{4.43}$$

$$\frac{\partial\,\Pi^{23}}{\partial\,y}+\frac{\partial\,\Pi^{33}}{\partial\,z}=0\ , \tag{4.44}$$

and,

$$\frac{\partial\,B^{33}}{\partial\,y}-\frac{\partial\,B^{23}}{\partial\,z}=0\ , \tag{4.45}$$

$$\frac{\partial\,B^{23}}{\partial\,y}+\frac{\partial\,B^{33}}{\partial\,z}=0\ . \tag{4.46}$$

This appearance of the Cauchy–Riemann equations is to be expected when working with shear–free null hypersurfaces [8], [9].

Next we examine the integrability condition (4.9). To evaluate this equation we first need to calculate ${}^{(3)}\nabla_c\,s_{ab}$ and $\dot{s}_{ab}+\theta\,s_{ab}$. The only non–vanishing Christoffel symbols for the line element (2.1) are $\Gamma^1_{14}=\dot{A}/A$, $\Gamma^2_{24}=\dot{B}/B$, $\Gamma^3_{34}=\dot{C}/C$, $\Gamma^4_{11}=A\,\dot{A}$, $\Gamma^4_{22}=B\,\dot{B}$ and $\Gamma^4_{33}=C\,\dot{C}$. Using these and the fact that $B=C$ we find that

$${}^{(3)}\nabla_\beta\,s_{ab}=0\ ,\qquad {}^{(3)}\nabla_4\,s_{ab}=0\ , \tag{4.47}$$

where β takes values 1, 2, 3 and

$$\dot{s}_{ab}+\theta\,s_{ab}=\frac{\partial s_{ab}}{\partial\,t}+\frac{\dot{A}}{A}\,s_{ab}\ . \tag{4.48}$$

Then Eq. (4.9) yields the following:

$$\frac{1}{A}\,A_{1ab}+B_{ab}=\frac{1}{A^2}\left(\frac{\partial}{\partial\,x}+\frac{\partial}{\partial\,T}\right)(A\,s_{ab})\ , \tag{4.49}$$

with $T(t)$ introduced in (2.20), and

$$\begin{aligned}\frac{1}{A}\,A_{2ab}\ &=\ \frac{1}{A}\,\frac{\partial s_{ab}}{\partial\,y}+(\sigma_{(2)}-\sigma_{(1)})\{\,s_{22}\,(\delta^1_b\,\delta^2_a+\delta^1_a\,\delta^2_b)\\ &\quad +s_{23}\,(\delta^1_a\,\delta^3_b+\delta^1_b\,\delta^3_a)\}\ ,\end{aligned} \tag{4.50}$$

$$\begin{aligned}\frac{1}{A}\,A_{3ab}\ &=\ \frac{1}{A}\,\frac{\partial s_{ab}}{\partial\,z}+(\sigma_{(2)}-\sigma_{(1)})\{\,s_{23}\,(\delta^1_b\,\delta^2_a+\delta^1_a\,\delta^2_b)\\ &\quad +s_{33}\,(\delta^1_a\,\delta^3_b+\delta^1_b\,\delta^3_a)\}\ .\end{aligned} \tag{4.51}$$

Since A_{cab} is orthogonal to u^c on all its indices we also have,

$$A_{4ab}=0\,,\qquad A_{c4b}=0\,,\qquad A_{ca4}=0\ . \tag{4.52}$$

With these equations for A_{cab} it is easily shown that Eqs. (4.11), (4.17) and the last of (4.37) are identically satisfied.

We now turn our attention to (4.22) and (4.23). We first note that,

$$s^b{}_q \, {}^{(3)}\nabla_b \, \lambda_m = 0 \ , \tag{4.53}$$

and

$$ {}^{(3)}\nabla_a \, s^{ab} = 0 \ . \tag{4.54}$$

Then (4.23) reduces to,

$$(A_m{}^b{}_q - A_q{}^b{}_m)\,\lambda_b + (\sigma^b{}_q\, s_{mb} - \sigma^b{}_m\, s_{qb})\,\dot{\phi} = 0 \ . \tag{4.55}$$

For convenience we shall define,

$$X_{(b)mq} := A_m{}^b{}_q - A_q{}^b{}_m = -X_{(b)qm} \ . \tag{4.56}$$

This allows us to write (4.55) as,

$$X_{(b)mq}\,\lambda_b + (\sigma^b{}_q\, s_{mb} - \sigma^b{}_m\, s_{qb})\,\dot{\phi} = 0 \ . \tag{4.57}$$

After a simple calculation using (4.49)–(4.52) we find that the non–zero components of $X_{(b)mq}$ are $X_{(2)12} = -\,X_{(3)13}$ and $X_{(2)13} = -\,X_{(3)12}$ with

$$X_{(2)12} = \frac{1}{B^2}\left\{-A\,B_{22} + \frac{1}{A}\left(\frac{\partial}{\partial\, x} + \frac{\partial}{\partial\, T}\right)(A\,s_{22}) - A\,(\sigma_{(2)} - \sigma_{(1)})\,s_{22}\right\} \ , \tag{4.58}$$

and

$$X_{(2)13} = \frac{1}{B^2}\left\{-A\,B_{23} + \frac{1}{A}\left(\frac{\partial}{\partial\, x} + \frac{\partial}{\partial\, T}\right)(A\,s_{23}) - A\,(\sigma_{(2)} - \sigma_{(1)})\,s_{23}\right\} \ , \tag{4.59}$$

As a result of this $X_{(b)mq}\,\lambda_b = 0$ and (4.57) reduces to

$$\sigma^b{}_q\, s_{mb} - \sigma^b{}_m\, s_{qb} = 0 \ . \tag{4.60}$$

This is clearly an identity since $\sigma_{(2)} = \sigma_{(3)}$ and $s_{22} = -s_{33}$, $s_{23} = s_{32}$ are the only non–zero components of s_{ab}. Thus we have now shown that (4.23) is identically satisfied. That (4.22) is also identically satisfied can be seen as follows: B^{ab} and Π^{ab} satisfy the same equations as s^{ab} and so we have,

$$\sigma^a{}_m\,\Pi_{aq} - \sigma^a{}_q\,\Pi_{am} = 0 \tag{4.61}$$

$$\sigma^a{}_m\,B_{aq} - \sigma^a{}_q\,B_{am} = 0 \ . \tag{4.62}$$

Then Eq. (4.22) becomes

$${}^{(3)}\nabla_b\,(A_m{}^b{}_q - A_q{}^b{}_m) = 0 \ , \tag{4.63}$$

which we write as

$$X_{(c)mq;c} + \Gamma^d_{mc}\,X_{(c)qd} + \Gamma^c_{dc}\,X_{(d)mq} + \Gamma^d_{qc}\,X_{(c)dm} = 0\ , \tag{4.64}$$

where $X_{(c)mq}$ is given by (4.56). Substituting for $X_{(c)mq}$ from (4.58)–(4.59) and using the Christoffel symbols which were listed after (4.46) we find that this equation is an identity.

At this point the equations we have left to satisfy are (4.4), (4.29), and (4.42)–(4.46). As we mentioned previously (4.29) is extremely complicated and to simplify the calculations which follow it is convenient to make use of the null tetrad in the background space–time which was introduced in (2.22). In terms of this tetrad s^{ab} can be written (because $s^a{}_a = 0$, $s^{ab}\,u_b = 0$, $s^{ab}\,k_b = 0$ and so $s^{ab}\,l_b = 0$)

$$s^{ab} = \bar{s}\,m^a\,m^b + s\,\bar{m}^a\,\bar{m}^b\ , \tag{4.65}$$

with

$$\bar{s} = -\,B^2\,(\alpha + i\,\beta)\ . \tag{4.66}$$

Since Π^{ab} and B^{ab} also satisfy the conditions given immediately before (4.65) we can write,

$$\Pi^{ab} = \bar{\Pi}\,m^a\,m^b + \Pi\,\bar{m}^a\,\bar{m}^b\ , \tag{4.67}$$

with

$$\bar{\Pi} = -\,B^2\,(\Pi^{33} + i\,\Pi^{23})\ , \tag{4.68}$$

and

$$B^{ab} = \bar{\mathcal{B}}\,m^a\,m^b + \mathcal{B}\,\bar{m}^a\,\bar{m}^b\ , \tag{4.69}$$

where

$$\bar{\mathcal{B}} = -\,B^2(B^{33} + i\,B^{23})\ . \tag{4.70}$$

We recall that $\sigma_{ab}\,m^a\,m^b$, which was first introduced in (4.34), is a gauge–invariant first order small variable and we can write it as

$$\sigma_{ab}\,m^a\,m^b = \kappa\,F(\phi)\ , \tag{4.71}$$

for some function κ. Multiplying the second of (4.1) by $m^a\,m^b$ and replacing $\sigma_{ab}\,m^a\,m^b$ by the right–hand side of (4.71) we find (after noting that $\dot{m}^a = 0$ since $m^a{}_{;d} = \dot{B}\,B^{-1}\,u^a\,m_d$ and m_a is orthogonal to u^a)that

$$\kappa = s\ , \tag{4.72}$$

and

$$\mathcal{B} = \dot{s} + \theta\,s\ . \tag{4.73}$$

We now have an equation for $\mathcal{B}$ in terms of s but for clarity we shall refrain from replacing $\mathcal{B}$ in the equations which follow by the right–hand side of this equation until the end. Repeating the above calculation on the first of (4.1) yields

$$A_{cab}\, m^a\, m^b = {}^{(3)}\nabla_c\, s \ , \tag{4.74}$$

which is identically satisfied as a consequence of (4.49)–(4.52) and (4.73).

We also need to express $\tilde{E}^{ab}$ and H^{ab} on this tetrad. It follows from (4.21) and the expressions for s^{ab}, B^{ab} and Π^{ab} given above that

$$\tilde{E}^{ab} = \bar{\tilde{E}}\, m^a\, m^b + \tilde{E}\, \bar{m}^a\, \bar{m}^b \ , \tag{4.75}$$

where

$$\bar{\tilde{E}} = \left(\frac{1}{2}\,\bar{\Pi} - \bar{\mathcal{B}}\right)\, F + \left(\frac{1}{A}\,\bar{s}\right)\, F' \ . \tag{4.76}$$

We note in passing that, making use of (4.71)–(4.72) and (4.74)–(4.75), we can write E'_{ab} in (4.36) as

$$E'_{ab} = \bar{E}'\, m^a\, m^b + E'\, \bar{m}^a\, \bar{m}^b \ , \tag{4.77}$$

with

$$\bar{E}' = \left(\frac{1}{2}\,\bar{\Pi} - \bar{\mathcal{B}} + \frac{\dot{A}}{A}\,\bar{s}\right)\, F + \frac{1}{A}\,\bar{s}\, F' \ . \tag{4.78}$$

Now we consider H^{ab}. First, substituting for A_{cab} from Eqs. (4.49)–(4.52) in (4.12) we observe that $H^{ab} = 0$ except for

$$\begin{aligned} H^{22} = -H^{33} \ &= \ \left\{(\sigma_{(2)} - \sigma_{(1)})\, s^{23} + B^{23} - \frac{1}{A^2\, B^4}\, \left(\frac{\partial}{\partial\, x} + \frac{\partial}{\partial\, T}\right)\, (A\, s_{23})\right\}\, F \\ &\quad -\frac{1}{A}\, s^{23}\, F' \ , \end{aligned} \tag{4.79}$$

and

$$\begin{aligned} H^{23} = H^{32} \ &= \ \left\{(\sigma_{(2)} - \sigma_{(1)})\, s^{33} - B^{22} - \frac{1}{A^2\, B^4}\, \left(\frac{\partial}{\partial\, x} + \frac{\partial}{\partial\, T}\right)\, (A\, s_{33})\right\}\, F \\ &\quad +\frac{1}{A}\, s^{22}\, F' \ , \end{aligned} \tag{4.80}$$

where $\partial/\partial\, T = A(\partial/\partial\, t)$. Then in terms of the null tetrad we can write

$$H^{ab} = \bar{H}\, m^a\, m^b + H\, \bar{m}^a\, \bar{m}^b \ , \tag{4.81}$$

with

$$\bar{H} = i\left\{(\sigma_{(2)}-\sigma_{(1)})\,\bar{s}+\bar{\mathcal{B}}-\frac{1}{A^2\,B^2}\left(\frac{\partial}{\partial\,x}+\frac{\partial}{\partial\,T}\right)(A\,B^2\,\bar{s})\right\}F \\ -i\,\frac{1}{A}\,\bar{s}\,F'\ . \tag{4.82}$$

We note that since s^{ab}, B^{ab} and Π^{ab} satisfy the Cauchy–Riemann equations, $\bar{s}$, $\bar{\mathcal{B}}$ and $\bar{\Pi}$ are independent of $\bar{\zeta}$. But $\bar{H}$ and $\bar{\tilde{E}}$ are derived from these quantities and thus they are also independent of $\bar{\zeta}$.

Next we wish to express W^{bt} given by (3.28) on the null tetrad but first we need to examine $h_a^{(b}\,\eta^{t)rsd}\,u_r\,H^a{}_{s;d}$. Using (4.81) and recalling that $m^a{}_{;\,d}=\dot{B}\,B^{-1}\,u^a\,m_d$ we deduce that

$$h^b_a\,\eta^{trsd}\,u_r\,H^a{}_{s;d}=\eta^{trsd}\,u_r\,\bar{H}_{,d}\,m^b\,m_s+\eta^{trsd}\,u_r\,H_{,d}\,\bar{m}^b\,\bar{m}_s\ . \tag{4.83}$$

Taking into account that $\partial\,H/\partial\,\zeta=0$ we find that, with respect to the null tetrad, we can write

$$H_{,d}=\sqrt{2}\,B^{-1}\,H_{\bar{\zeta}}\,\bar{m}_d-A^{-1}\,(H_x+H_T)\,u_d+H_x\,\phi_{,\,d}\ , \tag{4.84}$$

where $H_{\bar{\zeta}}=\partial\,H/\partial\,\bar{\zeta}$, $H_x=\partial\,H/\partial\,x$ and $H_T=\partial\,H/\partial\,T$. Putting this into (4.83) yields

$$h^b_a\,\eta^{trsd}\,u_r\,H^a{}_{s;d}=\frac{i}{A}\,(\bar{H}_x\,m^b\,m^t-H_x\,\bar{m}^b\,\bar{m}^t)\ , \tag{4.85}$$

which is symmetric in b and t. We are now in a position to write W^{bt} in terms of the null tetrad. We find it convenient to introduce a new variable $\mathcal{E}$ which is defined by

$$\tilde{E}^{bt}+\frac{1}{2}\,\Pi^{bt}=\mathcal{E}\,\bar{m}^b\,\bar{m}^t+\bar{\mathcal{E}}\,m^b\,m^t\ . \tag{4.86}$$

It is easily seen (using (4.67) and (4.76)) that

$$\mathcal{E}=(\Pi-\mathcal{B})\,F+A^{-1}\,s\,F'\ . \tag{4.87}$$

Then replacing $E^{bt}+\frac{1}{2}\Pi^{bt}$ and $h_a^{(b}\,\eta^{t)rsd}\,u_r\,H^a{}_{s;d}$ in (3.28) by (4.85) and (4.86) respectively gives (since ${}^{(3)}\nabla_c\,m^a=0$ and $\dot{m}^a=0$)

$$W^{bt} = \left(-\dot{\bar{\mathcal{E}}}-\frac{2}{3}\,\theta\,\mathcal{E}-i\,A^{-1}\,\bar{H}_x\right)m^b\,m^t+\bar{\mathcal{E}}\,m_f\,m^{(t}\,\sigma^{b)f} \\ + \left(-\dot{\mathcal{E}}-\frac{2}{3}\,\theta\,\mathcal{E}+i\,A^{-1}\,H_x\right)\bar{m}^b\,\bar{m}^t+\mathcal{E}\,\bar{m}_f\,\bar{m}^{(t}\,\sigma^{b)f}\ . \tag{4.88}$$

But a simple calculation shows that

$$m_f\, m^{(t}\,\sigma^{b)f} = \sigma_{(2)}\, m^b\, m^t = \left(\frac{\dot{B}}{B} - \frac{1}{3}\,\theta\right) m^b\, m^t \ , \tag{4.89}$$

and thus we can write,

$$W^{bt} = \bar{W}\, m^b\, m^t + W\, \bar{m}^b\, \bar{m}^t \ , \tag{4.90}$$

where

$$\bar{W} = -\frac{1}{A\,B}\,\frac{\partial}{\partial\, t}(A\,B\,\bar{\mathcal{E}}) - \frac{i}{A}\,\bar{H}_x \ . \tag{4.91}$$

Substituting from (4.82) and (4.87) reveals,

$$\bar{W} = P\,F + Q\,F' \ , \tag{4.92}$$

with

$$\begin{aligned} P \ &= \ -\frac{1}{AB}\,\frac{\partial}{\partial\, t}\{A\,B\,(-\bar{\mathcal{B}} + \bar{\Pi})\} + \frac{1}{A}\,(\sigma_{(2)} - \sigma_{(1)})\,\bar{s}_x \\ &\quad +\frac{1}{A}\,\bar{\mathcal{B}}_x - \frac{1}{A^3B^2}\left(\frac{\partial}{\partial\, x} + \frac{\partial}{\partial\, T}\right)(A\,B^2\,\bar{s}_x) \ , \end{aligned} \tag{4.93}$$

and

$$\begin{aligned} Q \ &= \ \frac{1}{A}\,(\sigma_{(2)} - \sigma_{(1)})\,\bar{s} - \frac{1}{A^3B^2}\left(\frac{\partial}{\partial\, x} + \frac{\partial}{\partial\, T}\right)(A\,B^2\,\bar{s}) \\ &\quad -\frac{1}{A^2}\,\bar{s}_x + \frac{1}{A}\,\bar{\Pi} - \frac{1}{AB}\,\frac{\partial}{\partial\, t}(B\,\bar{s}) \ . \end{aligned} \tag{4.94}$$

We now turn our attention to Eq.(4.29). We first observe that since $\sigma_{(2)} = \sigma_{(3)}$, $\sigma_{(1)} + 2\,\sigma_{(2)} = 0$, and $s^{22} = -s^{33}$, $s^{32} = s^{23}$ are the only non–zero components of s^{ab}, the coefficient of F' in Eq.(4.29) vanishes identically. Therefore

$$\begin{aligned} {}^{(3)}\nabla_c\, W^{bt} \ &= \ 3\,(-\,\sigma^2\, A_c{}^{bt} + \sigma^b{}_f\,\sigma^f{}_s\, A_c{}^{st} + \sigma^b{}_f\,\sigma^{st}\, A_c{}^f{}_s \\ &\quad +\sigma^f{}_s\,\sigma^{st}\, A_c{}^b{}_f - h^{bt}\,\sigma_a{}^d\,\sigma^{af}\, A_{cfd})\, F \ . \end{aligned} \tag{4.95}$$

Using (4.49)–(4.52) in this we find, after a lengthy calculation, that the coefficient of F on the right hand side vanishes and so we are left with

$${}^{(3)}\nabla_c\, W^{bt} = 0 \ . \tag{4.96}$$

Thus from (4.90) (since ${}^{(3)}\nabla_c m^a = 0$) we have

$${}^{(3)}\nabla_c\, \bar{W} = 0 \ , \tag{4.97}$$

which implies that

$$\frac{\partial\bar{W}}{\partial x}=0\ ,\qquad \frac{\partial\bar{W}}{\partial\zeta}=0\ ,\qquad \frac{\partial\bar{W}}{\partial\bar{\zeta}}=0\ . \tag{4.98}$$

Substitution of $\bar{W}$ from (4.92) into the first of (4.98) and using the arbitrariness of $F(\phi)$ we find that we must have $P=0$ and $Q=0$. Hence $\bar{W}=0$ and the other two equations in (4.98) are identically satisfied. We find it convenient now to write,

$$\bar{s}=-\frac{1}{A}\,\mathcal{G}(\zeta,\,x,\,t)\ , \tag{4.99}$$

where $\mathcal{G}$ is an analytic function of ζ. This follows from (4.42) and (4.66). Then from (4.94) with $Q=0$ we obtain *an equation for* $\bar{\Pi}$, namely,

$$\bar{\Pi}=-\frac{2}{A^2}\left(D\,\mathcal{G}+A\,\frac{\dot{B}}{B}\,\mathcal{G}\right)\ . \tag{4.100}$$

Here the operator D is given by $D=\partial/\partial\,x+A\,\partial/\partial\,t=\partial/\partial\,x+\partial/\partial\,T$. It follows from this and (4.68) that $\Pi^{33}+i\,\Pi^{23}$ is analytic in ζ and so (4.43) and (4.44) are automatically satisfied.

We now consider (4.93) with $P=0$. Substituting for $\bar{\Pi}$ and $\bar{\mathcal{B}}$ from (4.73) and (4.100) respectively results in the following *wave–equation for* $\mathcal{G}(\zeta\ ,x\ ,t)$:

$$D^2\,\mathcal{G}+A\left(\frac{\dot{B}}{B}-\frac{\dot{A}}{A}\right)D\,\mathcal{G}=0\ . \tag{4.101}$$

We note that we have made use of the field equations (2.3)–(2.5) (with $B=C$) to write the equation in this form.

The only equation remaining to be satisfied is (4.4). We shall first examine the case when $a=1$. Using (4.49)–(4.52) and recalling the only non–zero Christoffel symbols for the line–element (2.1) with $B=C$ given prior to Eq.(4.47) above, we find that (4.4) with $a=1$ reduces to

$$(\sigma_{(2)}-\sigma_{(1)})\left(\frac{\partial}{\partial\,t}(A\,s_{ab})+\frac{\partial\,s_{ab}}{\partial\,x}-2\,A\frac{\dot{B}}{B}\,s_{ab}\right)=0\ . \tag{4.102}$$

Multiplying this equation by $m^a\,m^b$ and noting that $m^a{}_{,d}=\dot{B}\,B^{-1}\,m^a\,u_d$ yields

$$(\sigma_{(2)}-\sigma_{(1)})D\,\mathcal{G}=0\ . \tag{4.103}$$

It is easily checked that Eq.(4.103) is equivalent to Eq.(4.102). We are interested here in an anisotropic universe and for this reason we cannot have

$\sigma_{(2)} = \sigma_{(1)}$ (we already have $\sigma_{(2)} = \sigma_{(3)}$ and if we also put $\sigma_{(2)} = \sigma_{(1)}$ the result is an isotropic universe). Thus (4.103) gives us a remarkably simple first order wave–equation,

$$D\,\mathcal{G} = 0\ , \tag{4.104}$$

from which we can conclude that $\mathcal{G} = \mathcal{G}(\zeta\ , x - T(t))$. Now (4.101) is automatically satisfied. We note that if the background cosmology is isotropic then the line–element (2.1) is simplified to $A = B = C = t^{2/3} = R(t)$ (say). In this case $\sigma_{(2)} = \sigma_{(1)}$ and so (4.103) is satisfied while the wave–equation for $\mathcal{G}(\zeta\ , x\ , t)$ is (4.101) specialised to

$$D^2\mathcal{G} = 0\ , \tag{4.105}$$

and (4.100) becomes

$$\bar{\Pi} = -\frac{2}{R^2}\left(D\mathcal{G} + \dot{R}\,\mathcal{G}\right)\ . \tag{4.106}$$

These are precisely the equations (5.37) and (5.36) found in [1] when our general formulation for studying shear–free gravitational radiation in isotropic cosmologies is specialised to apply to perturbations of the Einstein–de Sitter universe.

We have yet to examine (4.4) when $a = 2$, 3 or 4. The case when $a = 4$ is identically satisfied as a consequence of (4.52). To check that (4.4) is satisfied when $a = 2$ or $a = 3$ we first multiply across by $m^a\,m^b$. Noting that ${}^{(3)}\nabla_c\,m^a = 0$ this yields

$$\begin{aligned} {}^{(3)}\nabla_c\,\mathcal{B} \;&=\; \dot{A}_{cab}\,m^a\,m^b - 2\,\sigma^f{}_b\,A_{fac}\,m^a\,m^b + 2\,\sigma^f{}_b\,A_{afc}\,m^a\,m^b \\ &\quad + \sigma^f{}_c\,A_{fab}\,m^a\,m^b + \frac{4}{3}\,\theta\,A_{cab}\,m^a\,m^b\ . \end{aligned} \tag{4.107}$$

with $\mathcal{B}$ given by Eq.(4.70). Then using

$$\frac{\partial}{\partial t}(A_{cab}\,m^a\,m^b) = \left(\frac{\partial\,A_{cab}}{\partial\,t}\right)m^a\,m^b - 2\,\frac{\dot{B}}{B}\,A_{cab}\,m^a\,m^b\ , \tag{4.108}$$

we find after a straightforward calculation that (4.107) is identically satisfied.

Now that s^{ab}, Π^{ab} and B^{ab} are known we can calculate E' and H and form the gauge–invariant electric and magnetic parts of the perturbed Weyl tensor which are given by (4.71) and (4.81) respectively. We can write the result compactly as

$$E'_{ab} + i\,H_{ab} = -2\left\{\left(\frac{\dot{A}}{A^2} - \frac{\dot{B}}{AB}\right)\mathcal{G} + \frac{1}{A^2}\,\frac{\partial}{\partial\,x}(\mathcal{G}\,F)\right\}m_a\,m_b\ . \tag{4.109}$$

The scalar product of this complex–valued tensor with itself clearly vanishes and this is the algebraic property one expects the gauge–invariant part of the perturbed Weyl tensor to have if the perturbations describe gravitational waves [cf. Eq.(5.42) in [1] to which Eq.(4.109) specialises when the background cosmology is Einstein–de Sitter].

5 Discussion

The feature that makes this study of shear–free gravitational radiation propagating through a Bianchi I anisotropic universe so much more complicated than the corresponding study when the universe is isotropic [1] is the unavailability to us of the full perturbed shear of the matter world–lines as an Ellis–Bruni variable. This is in turn due to the central role played by the perturbed shear of the matter world–lines along with the perturbed anisotropic stress of the matter distribution in the study of gravitational waves. Nevertheless this paper demonstrates that these difficulties can be circumvented and the final result, which is summarised in the perturbed anisotropic stress (4.100) and the gauge–invariant perturbation of the Weyl tensor (4.107), with the function $\mathcal{G}$ involved in both satisfying (4.101) and (4.102), is easily surveyable. These perturbed quantities have been calculated with the aid of the basic Ellis–Bruni variables listed in (4.1). This is because A_{cab}, G_{cab}, B_{ab}, C_{ab} and Π_{ab} appearing in (4.1) are now all known in terms of $\mathcal{G}$: with s_{ab} given by (4.65) and (4.99) we have C_{ab} and G_{cab} given by (4.5) and (4.6). In addition B_{ab} is given by (4.69), (4.73) and (4.99) while A_{cab} is given by (4.50)–(4.52) and Π_{ab} is obtained from (4.67) and (4.100). At this level the most significant difference between the perturbations of the anisotropic universe compared to those of the isotropic universe is that in the former case the final wave equation to be satisfied by $\mathcal{G}$ is *first order* whereas it is *second order* in the latter case. It is clear from (4.103) that the anisotropy in the shear of the matter world–lines of the background is directly responsible for this.

Finally we see from (2.23) that the null hypersurfaces ϕ = constant in the anisotropic background have null geodesic generators *with shear* provided $\sigma_{(2)} \neq \sigma_{(3)}$. It is informative to carry out an analysis in parallel with that of section IV in this case. The calculations are briefly outlined in Appendix B. The significant conclusion is that the gauge–invariant part of the perturbed Weyl tensor does *not* possess all of the algebraic properties associated with gravitational waves having propagation direction $\phi_{,a}$ in the anisotropic background. For example, we cannot have $H^{ab}\,\phi_{,b}$ vanishing if the null hypersurfaces $\phi(x^a)$ = constant we are working with have shear. Therefore we cannot

interpret the perturbations as describing gravitational waves in this case. This observation suggests to us the possibility of constructing a *non–vacuum* generalisation of the Goldberg–Kerr theorem [10] which will also generalise the theorem of Szekeres [11] expressing the non–existence of Petrov Type N dust–filled universes. This will be discussed on another occasion.

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A Notation and Useful Equations

Throughout this paper we use the notation and sign conventions of [6] and we take the speed of light to be unity. We consider a four dimensional space–time manifold with metric tensor components g_{ab}, in a local coordinate system $\{x^a\}$, and a preferred congruence of world–lines tangent to the unit time–like vector field with components u^a (with $u^a\,u_a = -1$). With respect to this 4–velocity field the energy–momentum–stress tensor T^{ab} can be decomposed as

$$T^{ab} = \mu\, u^a\, u^b + p\, h^{ab} + q^a\, u^b + q^b\, u^a + \pi^{ab}\ , \tag{A.1}$$

where

$$h^{ab} = g^{ab} + u^a\, u^b\ , \tag{A.2}$$

is the projection tensor and

$$q^a\, u_a = 0, \qquad \pi^{ab}\, u_a = 0, \qquad \pi^a{}_a = 0\ . \tag{A.3}$$

Then μ is interpreted as the total energy density measured by an observer with 4–velocity u^a, q^a is the energy flow (such as heat flow) measured by this observer, p is the isotropic pressure and $\pi^a{}_b$ is the trace–free anisotropic stress (due, for example, to viscosity). After absorbing the coupling constant into the energy–momentum–stress tensor Einstein's field equations can be written as

$$R_{ab} - \frac{1}{2}\, g_{ab} R = T_{ab}\ , \tag{A.4}$$

where $R_{ab} = R_a{}^c{}_{bc}$ are the components of the Ricci tensor and R is the Ricci scalar.

We indicate covariant differentiation with a semicolon, partial differentiation by a comma and covariant differentiation in the direction of u^a by a dot. As usual square brackets denote skew–symmetrisation and round brackets denote symmetrisation. Thus the 4–acceleration of the time–like congruence is

$$\dot{u}^a := u^a{}_{;b}\, u^b. \tag{A.5}$$

With respect to u^a and h_{ab}, $u_{a;b}$ can be decomposed into

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\,\theta\, h_{ab} - \dot{u}_a\, u_b\ , \tag{A.6}$$

where

$$\omega_{ab} := u_{[a;b]} + \dot{u}_{[a}\, u_{b]}\ , \tag{A.7}$$

is the vorticity tensor of the congruence,

$$\sigma_{ab} := u_{(a;b)} + \dot{u}_{(a}\, u_{b)}\ , \tag{A.8}$$

is the shear tensor of the congruence and

$$\theta := u^a{}_{;a} \,, \tag{A.9}$$

is the expansion (or contraction) of the congruence.

Finally in this section we give the equations obtained when the Bianchi identities, written compactly as

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6}\, g^{c[a}\, R^{;b]} \tag{A.10}$$

are projected along and orthogonal to u^a. They are the *div–E equation*,

$$\begin{aligned}
&h^b_g\, E^{gd}{}_{;f}\, h^f_d + 3\,\omega^s\, H^b_s - \eta^{bapq}\, u_a\, \sigma^d{}_p\, H_{qd} = \frac{1}{3}\, h^b_c\, \mu^{;c} \\
&+\frac{1}{2}\left\{-\pi^{bd}{}_{;d} + u^b\, \sigma_{cd}\pi^{cd} - 3\,\omega^{bd} q_d + \sigma^{bd}\, q_d - \frac{2}{3}\,\theta\, q^b + \pi^{bd}\dot{u}_d\right\},
\end{aligned} \tag{A.11}$$

the *div–H equation*,

$$\begin{aligned}
&h^b_g\, H^{gd}{}_{;f}\, h^f_d - 3\,\omega^s\, E^b_s + \eta^{bapq}\, u_a\, \sigma^d{}_p\, E_{qd} = (\mu + p)\,\omega^b \\
&+\frac{1}{2}\,\eta^b{}_{qac}\, u^q\, q^{a;c} + \frac{1}{2}\,\eta^b{}_{qac}\, u^q\,(\omega^{dc} + \sigma^{dc})\,\pi^a{}_d \,,
\end{aligned} \tag{A.12}$$

the $\dot{E}$*–equation*,

$$\begin{aligned}
&h^b_f\, \dot{E}^{fg}\, h^t_g + h^{(b}_a\, \eta^{t)rsd}\, u_r H^a_{s;d} - 2\, H^{(b}_s\, \eta^{t)drs}\, u_d \dot{u}_r \\
&-E^{(t}_s\, \omega^{b)s} - 3\, E^{(t}_s\, \sigma^{b)s} + h^{tb}\, E^{dp}\, \sigma_{dp} + \theta\, E^{bt} = -\frac{1}{2}\,(\mu + p)\,\sigma^{tb} \\
&-\frac{1}{6}\, h^{tb}\,\{\dot{\mu} + \theta\,(\mu + p)\} - \, q^{(b}\,\dot{u}^{t)} - \frac{1}{2}\, u^{(b}\,\dot{q}^{t)} - \frac{1}{2}\, q^{(t;b)} \\
&+\frac{1}{2}\{\omega^{c(b} + \sigma^{c(b}\}\, u^{t)}\, q_c + \frac{1}{6}\,\theta\, u^{(t}\, q^{b)} - \frac{1}{2}\,\dot{\pi}^{bt} + \pi^{c(b}\, u^{t)}\,\dot{u}_c \\
&-\frac{1}{2}\,\{\omega^{c(b} + \sigma^{c(b}\}\,\pi^{t)}{}_c - \frac{1}{6}\,\theta\,\pi^{bt} \,,
\end{aligned} \tag{A.13}$$

and the $\dot{H}$*–equation*,

$$\begin{aligned}
&h^b_f\, \dot{H}^{fg}\, h^t_g - h^{(b}_a\, \eta^{t)rsd}\, u_r E^a_{s;d} + 2\, E^{(b}_s\, \eta^{t)drs}\, u_d \dot{u}_r \\
&-H^{(t}_s\, \omega^{b)s} - 3\, H^{(t}_s\, \sigma^{b)s} + h^{tb}\, H^{dp}\, \sigma_{dp} + \theta\, H^{bt} = -q^{(t}\,\omega^{b)} \\
&-\frac{1}{2}\,\eta^{(t}{}_{rad}\,\{\omega^{b)d} + \sigma^{b)d}\}\, u^r\, q^a - \frac{1}{2}\eta^{(b}{}_{rad}\,\pi^{t)a;d}\, u_r \\
&+\frac{1}{2}\eta^{(b}{}_{rad}\, u^{t)}\, u^r\,\{\omega^{cd} + \sigma^{cd}\}\,\pi^a{}_c \,.
\end{aligned} \tag{A.14}$$

B Null Hypersurfaces with Shear

We note from (2.23) that the null hypersurfaces ϕ = constant have null geodesic generators *with shear* provided $\sigma_{(2)} \neq \sigma_{(3)}$. We briefly describe in this appendix the perturbations analogous to those of Section IV in this case. Equations (4.1)–(4.32) still apply and we shall again assume that $B^{ab}\,\phi_{,b} = 0$ (the reason for this is given before (4.37)). Now recalling that s^{ab} is orthogonal to u_a and trace–free it follows from (4.10) and (4.18) that s^{ab} has only one independent component $s^{23} = s^{32}$. B^{ab} and Π^{ab} are also orthogonal to u_a and trace–free and satisfy the same equations as s^{ab}. Thus they also have only one independent component, $B^{23} = B^{32}$ and $\Pi^{23} = \Pi^{32}$. On account of this, the Cauchy–Riemann equations (4.40)–(4.46) now read,

$$\frac{\partial s^{23}}{\partial y} = 0\ , \qquad \frac{\partial s^{23}}{\partial z} = 0\ , \tag{B.1}$$

$$\frac{\partial \Pi^{23}}{\partial y} = 0\ , \qquad \frac{\partial \Pi^{23}}{\partial z} = 0\ , \tag{B.2}$$

$$\frac{\partial B^{23}}{\partial y} = 0\ , \qquad \frac{\partial B^{23}}{\partial z} = 0\ . \tag{B.3}$$

In the present context ($\sigma_{(2)} \neq \sigma_{(3)}$) equations (4.49)–(4.52) which we derived from (4.9) are modified to:

$$\begin{aligned} A_{1ab} &= (\delta^2_a\,\delta^3_b + \delta^3_a\,\delta^2_b)\left\{-A\,B_{23} + \frac{\partial s_{23}}{\partial x} + \dot{A}\,s_{23} + A\,\frac{\partial s_{23}}{\partial t}\right\}\ , &(\text{B.4})\\ A_{2ab} &= A\,s_{23}(\sigma_{(3)} - \sigma_{(1)})(\delta^1_a\,\delta^3_b + \delta^3_a\,\delta^1_b)\ , &(\text{B.5})\\ A_{3ab} &= A\,s_{23}(\sigma_{(2)} - \sigma_{(1)})(\delta^1_a\,\delta^2_b + \delta^2_a\,\delta^1_b)\ , &(\text{B.6})\end{aligned}$$

and

$$A_{4ab} = 0\ , \qquad A_{a4b} = 0\ , \qquad A_{ab4} = 0\ . \tag{B.7}$$

It follows immediately from these equations that (4.11) and (4.17) are identically satisfied. Using these equations in (4.12)–(4.14) we find that $H_{ab} = 0$ except for

$$H_{11} = \frac{A^2}{BC}\,s_{23}\left(\frac{\dot{C}}{C} - \frac{\dot{B}}{B}\right)F\ , \tag{B.8}$$

$$H_{22} = \left\{\left(\frac{\dot{B}}{C} - \frac{\dot{A}B}{AC}\right)s_{23} + \frac{B}{C}\,B_{23} - \frac{B}{A^2C}D(A\,s_{23})\right\}F - \frac{B}{AC}\,s_{23}\,F', \tag{B.9}$$

$$H_{33} = \left\{\left(\frac{\dot{A}C}{AB} - \frac{\dot{C}}{B}\right)s_{23} - \frac{C}{B}\,B_{23} + \frac{C}{A^2B}\,D(A\,s_{23})\right\}F + \frac{C}{AB}\,s_{23}F'\ . \tag{B.10}$$

Here the operator D is given by $D = \partial/\partial x + A\,\partial/\partial t$. We note that as a consequence of these equations we have

$$H^{ab}\,\phi_{,b} = \delta^1_a\,\frac{s_{23}}{B}\,(\sigma_{(3)} - \sigma_{(2)})\,F \neq 0\ . \tag{B.11}$$

Thus one of the algebraic properties of the gauge–invariant part of the perturbed Weyl tensor, namely $H^{ab}\phi_{,b} = 0$, which we expect to hold if the perturbations describe gravitational waves having the null hypersurfaces $\phi =$ constant as the histories of their wave–fronts, is not possible unless the null hypersurfaces are shear–free.

We now consider (3.28). First, a straightforward but tedious calculation gives,

$$h_{a(b}\,\eta_{t)}{}^{rsd}\,u_r\,H^a{}_{s;d} = \frac{1}{2}\,\frac{\partial}{\partial x}\left(\frac{B}{AC}\,H_{33} - \frac{C}{AB}\,H_{22}\right)(\delta^2_b\,\delta^3_t + \delta^3_b\,\delta^2_t)\ . \tag{B.12}$$

Then substituting for $\tilde{E}_{ab}$ and H_{ab} from (4.21) and (B8)–(B10) respectively in (3.28) and making use of (B.12) yields

$$W_{bt} = (PF + QF')(\delta^2_b\,\delta^3_t + \delta^3_b\,\delta^2_t)\ , \tag{B.13}$$

where (after recalling that $\sigma_{(1)} + \sigma_{(2)} + \sigma_{(3)} = 0$)

$$\begin{aligned} P \;&=\; -\frac{\partial}{\partial t}(\Pi_{23} - B_{23}) + \left\{\frac{1}{2}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) - \frac{\dot{A}}{A}\right\}(\Pi_{23} - B_{23}) + \frac{1}{A}\frac{\partial B_{23}}{\partial x} \\ &\quad -\frac{1}{A^3}\,D\left(A\frac{\partial s_{23}}{\partial x}\right) - \frac{3}{2A}\,\sigma_{(1)}\,\frac{\partial s_{23}}{\partial x}\ , \end{aligned} \tag{B.14}$$

and

$$\begin{aligned} Q \;&=\; \frac{1}{A}\,\Pi_{23} - \frac{1}{A}\frac{\partial s_{23}}{\partial t} + \frac{1}{2A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)s_{23} - \frac{1}{A^3}\,D(As_{23}) \\ &\quad -\frac{3}{2A}\sigma_{(1)}\,s_{23} - \frac{1}{A^2}\frac{\partial s_{23}}{\partial x}\ . \end{aligned} \tag{B.15}$$

Remembering that the only non–zero components of the background σ_{ab} are $\sigma_{(1)} = \sigma^1{}_1$, $\sigma_{(2)} = \sigma^2{}_2$ and $\sigma_{(3)} = \sigma^3{}_3$, and substituting for A_{cab} from (B4)–(B7) we find that, in this case, the right–hand side of (4.29) vanishes identically. Thus we have

$${}^{(3)}\nabla_c\,W_{bt} = 0\ . \tag{B.16}$$

It is easily shown, using the Cauchy–Riemann equations and the non–zero Christoffel symbols given before (4.47) that this equation is identically satisfied when $c = 2, 3$ or 4. However when $c = 1$ it implies,

$$\frac{\partial W_{bt}}{\partial x} = 0\ . \tag{B.17}$$

It follows from this and (B.13) that $P = 0$ and $Q = 0$. Similar to the shear–free case we now find it convenient to define

$$s_{23} = -\frac{1}{A}\,\mathcal{G}(x,t)\ , \tag{B.18}$$

and

$$B_{23} = -\frac{1}{A}\,\mathcal{H}(x,t)\ . \tag{B.19}$$

Then from (B.15) with $Q = 0$ we obtain

$$\Pi_{23} = -\frac{2}{A^2}\,D\,\mathcal{G} + \frac{1}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\mathcal{G}\ . \tag{B.20}$$

Finally we examine Eq. (4.4) when the null geodesic generators of the hypersurfaces have shear (i.e. when $\sigma_{(2)} \neq \sigma_{(3)}$). When $c = 2$ or $c = 3$ we find, after a lengthy calculation utilising (B4)–(B7), that $\mathcal{H}$ in (B.19) is now given by

$$\mathcal{H} = 2\,\mathcal{G}_t + \frac{1}{A}\,\mathcal{G}_x - \left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\mathcal{G}\ , \tag{B.21}$$

where $\mathcal{G}_t = \partial\mathcal{G}/\partial t$ and $\mathcal{G}_x = \partial\mathcal{G}/\partial x$. We note that this equation is valid if and only if $\sigma_{(1)} \neq \sigma_{(2)}$ and $\sigma_{(1)} \neq \sigma_{(3)}$. Then substituting for A_{cab} and $\mathcal{H}$ from (B4)–(B7) and (B.21) respectively in (4.4) with $c = 1$ yields

$$\frac{1}{A^2}\,D^2\,\mathcal{G} - \frac{1}{A}\left(\frac{\dot{B}}{B} - \frac{\dot{C}}{C}\right)D\,\mathcal{G} - \left(\frac{\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{C}}{AC} - 3\frac{\dot{B}\dot{C}}{BC}\right)\mathcal{G} = 0\ , \tag{B.22}$$

where we have used the field equations (2.3)–(2.5) to simplify the coefficient of $\mathcal{G}$. Eqs. (B.20) and (B.21) allow us to write,

$$\Pi_{23} - B_{23} = -\frac{1}{A^2}\,\mathcal{G}_x\ , \tag{B.23}$$

and

$$\frac{1}{A}\,\frac{\partial B_{23}}{\partial x} = -\frac{2}{A^2}\,G_{tx} - \frac{1}{A^3}\,\mathcal{G}_{xx} + \frac{1}{A^2}\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right)\mathcal{G}_x\ . \tag{B.24}$$

Using these two equations it is a simple calculation to show that (B14) with $P = 0$ is identically satisfied. We note that (4.4) with $c = 4$ gives $0 = 0$.

The basic Ellis–Bruni variables listed in (4.1) are now determined in terms of $\mathcal{G}$, with $\mathcal{G}(x,t)$ given by (B.22).The only non–vanishing component of s_{ab} is given by (B.18) and of B_{ab} by (B.19) and (B.21), with the corresponding component of Π_{ab} given by (B.20). As in section IV, C_{ab} and G_{cab} are obtained from (4.5) and (4.6) while A_{cab} is given by (B.4)–(B.7) in this case.